数学及力学演習 L 第 9 回（12/17） 解答

問題 1

(1) $\nabla \log |\boldsymbol{r}|$

$$= \nabla \log \sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{2x}{x^2+y^2+z^2} \\ \frac{2y}{x^2+y^2+z^2} \\ \frac{2z}{x^2+y^2+z^2} \end{pmatrix}$$

$$= \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{\boldsymbol{r}}{|\boldsymbol{r}|^2}$$

(2) $\nabla \times \left(|\boldsymbol{r}|^2 \boldsymbol{r}\right)$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \left(x^2+y^2+z^2\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)z - \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)y \\ \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)x - \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)z \\ \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)y - \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)x \end{pmatrix}$$

$$= \begin{pmatrix} 2yz-2zy \\ 2zx-2xz \\ 2xy-2yx \end{pmatrix}$$

$$= \boldsymbol{0}$$

(3) $\mathit{\Delta} \log |\boldsymbol{r}|$

$$= \frac{1}{2}\mathit{\Delta} \log\left(x^2+y^2+z^2\right)$$

$$= \frac{\left(x^2+y^2+z^2\right) - x\cdot 2x}{\left(x^2+y^2+z^2\right)^2} + \frac{\left(x^2+y^2+z^2\right) - y\cdot 2y}{\left(x^2+y^2+z^2\right)^2} + \frac{\left(x^2+y^2+z^2\right) - z\cdot 2z}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{3\left(x^2+y^2+z^2\right) - 2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{1}{x^2+y^2+z^2}$$

$$= \frac{1}{|\boldsymbol{r}|^2}$$

(4) $\nabla \cdot \left(\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}$$

$$= \frac{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} - x\cdot\frac{n}{2}\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}\cdot 2x}{\left(x^2+y^2+z^2\right)^n} + \frac{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} - y\cdot\frac{n}{2}\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}\cdot 2y}{\left(x^2+y^2+z^2\right)^n} + \frac{\left(x^2+y^2+z^2\right)^{\frac{n}{2}} - z\cdot\frac{n}{2}\cdot\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}\cdot 2z}{\left(x^2+y^2+z^2\right)^n}$$

$$= \frac{3}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}} - \frac{n\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{3-n}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}$$

$$= \frac{3-n}{|\boldsymbol{r}|^n}$$

$n=3$ のとき 0 となる。

(5) $\nabla\left(\boldsymbol{a}\cdot\boldsymbol{r}\right)$

$$= \nabla\left(a_1x+a_2y+a_3z\right)$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$= \boldsymbol{a}$$

(6) $\nabla \times \left(\boldsymbol{a}\times\boldsymbol{r}\right)$

$$= \nabla \times \begin{pmatrix} a_2z-a_3y \\ a_3x-a_1z \\ a_1y-a_2x \end{pmatrix}$$

$$= \begin{pmatrix} a_1+a_1 \\ a_2+a_2 \\ a_3+a_3 \end{pmatrix}$$

$$= 2\boldsymbol{a}$$

問題 2

$\varDelta x$, $\varDelta y$ だけ動いたときの T の変化を考えると

$$\begin{aligned}\varDelta T &= T(x+\varDelta x, y+\varDelta y) - T(x,y)\\ &\approx \mathrm{grad}\,T \cdot \begin{pmatrix}\varDelta x\\ \varDelta y\end{pmatrix}\end{aligned}$$

$\mathrm{grad}\,T$ と $\begin{pmatrix}\varDelta x\\ \varDelta y\end{pmatrix}$ とが逆向きのとき内積は最小。

すなわち、虫は $\mathrm{grad}\,T$ と逆の方向に進んで行く。

$\mathrm{grad}\,T = \begin{pmatrix}-2x\\ -6y\end{pmatrix}$ より

$\begin{pmatrix}\varDelta x\\ \varDelta y\end{pmatrix} = k\begin{pmatrix}2x\\ 6y\end{pmatrix}$ $(k>0)$ ならよい。

これから微分方程式を立てると

$$\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix} = \begin{pmatrix}2k & 0\\ 0 & 6k\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x_0 e^{2kt}\\ y_0 e^{6kt}\end{pmatrix}$$

$$\Leftrightarrow y = \frac{y_0}{x_0^3}x^3$$

問題 3

(1) $\mathrm{div}\,\mathrm{rot}\,\boldsymbol{v}$

$$\begin{aligned}&= \nabla\cdot(\nabla\times\boldsymbol{v})\\ &= |\,\nabla\,\nabla\,\boldsymbol{v}\,|\\ &= (\nabla\times\nabla)\cdot\boldsymbol{v}\\ &= \begin{pmatrix}\frac{\partial}{\partial y}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial z}\\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial x}\end{pmatrix}\cdot\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix}\\ &= 0\end{aligned}$$

(2) $\mathrm{rot}\,\mathrm{grad}\,\phi$

$$\begin{aligned}&= \nabla\times(\nabla\phi)\\ &= \nabla\times\begin{pmatrix}\phi_x\\ \phi_y\\ \phi_z\end{pmatrix}\\ &= \begin{pmatrix}\phi_{zy}-\phi_{yz}\\ \phi_{xz}-\phi_{zx}\\ \phi_{yx}-\phi_{xy}\end{pmatrix}\\ &= \mathbf{0}\end{aligned}$$

(3) $\mathrm{grad}\,\mathrm{div}\,\boldsymbol{v} - \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{v}$

$$\begin{aligned}&\mathrm{grad}\,(v_{1x}+v_{2y}+v_{3z}) - \mathrm{rot}\begin{pmatrix}v_{3y}-v_{2z}\\ v_{1z}-v_{3x}\\ v_{2x}-v_{1y}\end{pmatrix}\\ &= \begin{pmatrix}v_{1xx}+v_{2yx}+v_{3zx}\\ v_{1xy}+v_{2yy}+v_{3zy}\\ v_{1xz}+v_{2yz}+v_{3zz}\end{pmatrix} - \begin{pmatrix}v_{2xy}-v_{1yy}-v_{1zz}-v_{3xz}\\ v_{3yz}-v_{2zz}-v_{2xx}-v_{1yx}\\ v_{1zx}-v_{3xx}-v_{3yy}-v_{2zy}\end{pmatrix}\\ &= \begin{pmatrix}v_{1xx}+v_{1yy}+v_{1zz}\\ v_{2xx}+v_{2yy}+v_{2zz}\\ v_{3xx}+v_{3yy}+v_{3zz}\end{pmatrix}\\ &= \frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}+\frac{\partial^2 v_3}{\partial z^2}\\ &= \boldsymbol{\varDelta v}\end{aligned}$$

(4) $\boldsymbol{v}\times\mathrm{rot}\,\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$

$$\begin{aligned}&= \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\times\begin{pmatrix}v_{3y}-v_{2z}\\ v_{1z}-v_{3x}\\ v_{2x}-v_{1y}\end{pmatrix} +\\ &\quad\left(v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+v_3\frac{\partial}{\partial z}\right)\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}\\ &= \begin{pmatrix}v_2(v_{2x}-v_{1y}) - v_3(v_{1z}-v_{3x})\\ v_3(v_{3y}-v_{2z}) - v_1(v_{2x}-v_{1y})\\ v_1(v_{1z}-v_{3x}) - v_2(v_{3y}-v_{2z})\end{pmatrix} +\\ &\quad\begin{pmatrix}v_1v_{1x}+v_2v_{1y}+v_3v_{1z}\\ v_1v_{2x}+v_2v_{2y}+v_3v_{2z}\\ v_1v_{3x}+v_2v_{3y}+v_3v_{3z}\end{pmatrix}\\ &= \begin{pmatrix}v_1v_{1x}+v_2v_{2x}+v_3v_{3x}\\ v_1v_{1y}+v_2v_{2y}+v_3v_{3y}\\ v_1v_{1z}+v_2v_{2z}+v_3v_{3z}\end{pmatrix}\\ &= \frac{1}{2}\,\mathrm{grad}\,\left(v_1^2+v_2^2+v_3^2\right)\\ &= \frac{1}{2}\,\mathrm{grad}\,|\boldsymbol{v}|^2\end{aligned}$$

問題 4

$$\begin{aligned}\varDelta \boldsymbol{H} &= \mathrm{grad}\,\mathrm{div}\boldsymbol{H} - \mathrm{rot}\,\mathrm{rot}\boldsymbol{H}\\ &= -\mathrm{rot}\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)\\ &= -\varepsilon_0\,\mathrm{rot}\,\frac{\partial \boldsymbol{E}}{\partial t}\\ &= -\varepsilon_0\,\frac{\partial}{\partial t}\,\mathrm{rot}\boldsymbol{E}\\ &= -\varepsilon_0\,\frac{\partial}{\partial t}\left(-\mu_0\,\frac{\partial \boldsymbol{H}}{\partial t}\right)\\ &= \varepsilon_0\,\mu_0\,\frac{\partial^2 \boldsymbol{H}}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\varDelta \boldsymbol{E} &= \mathrm{grad}\,\mathrm{div}\boldsymbol{E} - \mathrm{rot}\,\mathrm{rot}\boldsymbol{E}\\ &= -\mathrm{rot}\left(-\mu_0\,\frac{\partial \boldsymbol{H}}{\partial t}\right)\\ &= \mu_0\,\frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{H}\\ &= \mu_0\,\frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)\\ &= \varepsilon_0\,\mu_0\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2}\end{aligned}$$

問題 5

$$\begin{aligned}\frac{dE\left(t\right)}{dt} &= \frac{1}{2}\frac{d}{dt}\left|\frac{d\boldsymbol{r}}{dt}\right|^2 + \frac{df\left(\boldsymbol{r}\right)}{dt}\\ &= \frac{1}{2}\frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\cdot\frac{d\boldsymbol{r}}{dt}\right) + \mathrm{grad}f\left(\boldsymbol{r}\right)\cdot\frac{d\boldsymbol{r}}{dt}\\ &= \frac{d\boldsymbol{r}}{dt}\cdot\frac{d^2\boldsymbol{r}}{dt^2} + \mathrm{grad}f\left(\boldsymbol{r}\right)\cdot\frac{d\boldsymbol{r}}{dt}\\ &= \frac{d\boldsymbol{r}}{dt}\cdot\left\{\frac{d^2\boldsymbol{r}}{dt^2} + \mathrm{grad}f\left(\boldsymbol{r}\right)\right\}\\ &= 0\end{aligned}$$

問題 6

$$\begin{aligned}s &= \int_\alpha^t \sqrt{\left(-a\sin t\right)^2 + \left(a\cos t\right)^2 + \left(a\tan\theta\right)^2}dt\\ &= \int_\alpha^t a\sqrt{1+\tan^2\theta}dt\\ &= \int_\alpha^t \frac{a}{\cos\theta}dt\\ &= \frac{a}{\cos\theta}\left(t-\alpha\right)\end{aligned}$$

$\alpha = 0$ として $s = \dfrac{a}{\cos\theta}t$ とすると

$$\boldsymbol{r}\left(s\right) = \begin{pmatrix} a\cos\left(\frac{\cos\theta}{a}s\right)\\ a\sin\left(\frac{\cos\theta}{a}s\right)\\ a\frac{\cos\theta}{a}s\tan\theta\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{t}\left(s\right) &= \frac{d\boldsymbol{r}\left(s\right)}{ds}\\ &= \begin{pmatrix} -\cos\theta\sin\left(\frac{\cos\theta}{a}s\right)\\ \cos\theta\cos\left(\frac{\cos\theta}{a}s\right)\\ \sin\theta\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\frac{d\boldsymbol{t}\left(s\right)}{ds} &= \begin{pmatrix} -\frac{\cos^2\theta}{a}\cos\left(\frac{\cos\theta}{a}s\right)\\ -\frac{\cos^2\theta}{a}\sin\left(\frac{\sin\theta}{a}s\right)\\ 0\end{pmatrix}\\ &= \frac{\cos^2\theta}{a}\begin{pmatrix} -\cos\left(\frac{\cos\theta}{a}s\right)\\ -\sin\left(\frac{\cos\theta}{a}s\right)\\ 0\end{pmatrix}\end{aligned}$$

よって

$$\kappa = \frac{\cos^2\theta}{a}\,,\ \boldsymbol{n}\left(s\right) = \begin{pmatrix} -\cos\left(\frac{\cos\theta}{a}s\right)\\ -\sin\left(\frac{\cos\theta}{a}s\right)\\ 0\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{b}\left(s\right) &= \boldsymbol{t}\left(s\right)\times\boldsymbol{n}\left(s\right)\\ &= \begin{pmatrix} \sin\theta\,\sin\left(\frac{\cos\theta}{a}s\right)\\ -\sin\theta\,\cos\left(\frac{\cos\theta}{a}s\right)\\ \cos\theta\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tau &= -\left|\frac{d\boldsymbol{b}\left(s\right)}{ds}\right|\\ &= -\left|\begin{matrix} -\frac{\sin\theta\,\cos\theta}{a}\cos\left(\frac{\cos\theta}{a}s\right)\\ -\frac{\sin\theta\cos\theta}{a}\sin\left(\frac{\cos\theta}{a}s\right)\\ 0\end{matrix}\right|\\ &= -\left|\frac{\sin\theta\,\cos\theta}{a}\right|\end{aligned}$$

問題 7

(1) $\left\{\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{pmatrix} \cdot \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix}\right\}'$

$= \{a_1(t)\,b_1(t) + a_2(t)\,b_2(t) + a_3(t)\,b_3(t)\}'$

$= a_1'(t)\,b_1(t) + a_2'(t)\,b_2(t) + a_3'(t)\,b_3(t) +$

$\quad a_1(t)\,b_1'(t) + a_2(t)\,b_2'(t) + a_3(t)\,b_3'(t)$

$= \boldsymbol{a}'(t) \cdot \boldsymbol{b}(t) + \boldsymbol{a}(t) \cdot \boldsymbol{b}'(t)$

(2) $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$, $\boldsymbol{c}(t)$ が互いに直交しているので

$$\begin{pmatrix} \boldsymbol{a}'(t) \\ \boldsymbol{b}'(t) \\ \boldsymbol{c}'(t) \end{pmatrix} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}(t) \\ \boldsymbol{b}(t) \\ \boldsymbol{c}(t) \end{pmatrix}$$

と書き表すことができる。

（ただし $C_1 \sim C_9$ は t の関数）

また常に単位ベクトルであることから、

たとえば $\boldsymbol{a}(t)$ について

$\left(|\boldsymbol{a}(t)|^2\right)' = 0$

$\Leftrightarrow \{\boldsymbol{a}(t) \cdot \boldsymbol{a}(t)\}' = 0$

$\Leftrightarrow 2\boldsymbol{a}'(t) \cdot \boldsymbol{a}(t) = 0$

$\Leftrightarrow \boldsymbol{a}'(t) \cdot \boldsymbol{a}(t) = 0$

$\Leftrightarrow \{C_1\boldsymbol{a}(t) + C_2\boldsymbol{b}(t) + C_3\boldsymbol{c}(t)\} \cdot \boldsymbol{a}(t) = 0$

$\Leftrightarrow C_1 = 0$（直交性に注意）

同様にして $C_1 = C_5 = C_9 = 0$ がいえる。

(1) で証明したことから

$\boldsymbol{a}'(t) \cdot \boldsymbol{b}(t) + \boldsymbol{a}(t) \cdot \boldsymbol{b}'(t) = \{\boldsymbol{a}(t) \cdot \boldsymbol{b}(t)\}'$

$\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$ は直交しているので

$\boldsymbol{a}'(t) \cdot \boldsymbol{b}(t) + \boldsymbol{a}(t) \cdot \boldsymbol{b}'(t) = 0$

$\Leftrightarrow \{C_2\boldsymbol{b}(t) + C_3\boldsymbol{c}(t)\} \cdot \boldsymbol{b}(t) +$

$\quad \boldsymbol{a}(t) \cdot \{C_4\boldsymbol{a}(t) + C_6\boldsymbol{c}(t)\} = 0$

$\Leftrightarrow C_2 + C_4 = 0$

$\Leftrightarrow C_2 = -C_4 \equiv \lambda_3(t)$

以下同様にして

$-C_3 = C_7 \equiv \lambda_2(t)$

$C_6 = -C_8 \equiv \lambda_1(t)$

とおけばよい。